MATH 2230 A Midterm 2.

Problem 1: On C, $dz = ie^{i\theta}d\theta$

$$z^i = e^{i\operatorname{Log} z}$$
$$= e^{i\log e^{i\theta}}$$
$$= e^{-\theta}$$

$$\int_C z^i dz = \int_0^{\pi} e^{-\theta}\cdot ie^{i\theta}d\theta$$
$$= i\int_0^{\pi} e^{\theta(-1+i)}\,d\theta$$
$$= \frac{i}{-1+i}\left(e^{\pi(-1+i)}-1\right)$$
$$= \frac{1-i}{2}\left(-e^{-\pi}-1\right)$$

Problem 2: Case 1, if $z\in\{z\in\mathbb{C}\mid |z|>3\}$, then $\frac{s^5+3s+1}{(s-z)^3}$ is analytic $\forall s\in\{z\in\mathbb{C}\mid |z|<3\}$.

Hence by Cauchy-Goursat theorem, $\int_C \frac{s^5+3s+1}{(s-z)^3}ds = 0$

Case 2, if $z\in\{z\in\mathbb{C}\mid |z|<3\}$, let $f(s)=s^5+3s+1$

By generalized Cauchy integral formula,
$$f''(z) = \frac{2!}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^3}$$

Hence $\int_C \frac{s^5+3s+1}{(s-z)^3}ds = \pi i\left(20z^3\right)$

Problem 3: Consider $z(z^2+8)=0$

$z = 0, \pm 2\sqrt{2}i$

It can be seen that 0 and $2\sqrt{2}i$ is inside the circle C.

Therefore,
$$\int_C \frac{\cos z}{z(z^2+8)} dz = \int_C \frac{\cos z \, dz}{z(z-2\sqrt{2}i)(z+2\sqrt{2}i)}$$

$$= 2\pi i \left(\frac{1}{8} + \frac{\cos 2\sqrt{2}i}{2\sqrt{2}i(4\sqrt{2}i)} \right)$$

$$= 2\pi i \left(\frac{1}{8} - \frac{\cos 2\sqrt{2}i}{16} \right)$$

Problem 4:

i)
$$\left| \int_{Cir(0,\varepsilon)} \frac{f(z)}{z-z_0} dz \right| \le \int_{Cir(0,\varepsilon)} \frac{|f(z)|}{|z-z_0|} dz$$

$$\le \int_{Cir(0,\varepsilon)} \frac{\ln\left(\frac{1}{\varepsilon}\right)}{||z_0|-\varepsilon|} dz$$

$$= \frac{2\pi}{||z_0|-\varepsilon|} \cdot \varepsilon \ln\left(\frac{1}{\varepsilon}\right)$$

Hence $\lim_{\varepsilon \to 0} \int_{Cir(0,\varepsilon)} \frac{f(z)}{z-z_0} dz = 0$

ii) $\int_{Cir(0,1)} \frac{f(z)}{z-z_0} dz = \int_{Cir(0,\varepsilon)} \frac{f(z)}{z-z_0} dz + \int_{Cir(z_0,\varepsilon)} \frac{f(z)}{z-z_0} dz$

iii) By Cauchy Integral Formula, $\int_{Cir(z_0,\varepsilon)} \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$

Since $|f(z)| \le \ln\left(\frac{1}{|z|}\right)$ in $\mathfrak{D}$, if $|z|=1$, then $f(z)=0$,

we have $0 = \int_{Cir(0,\varepsilon)} \frac{f(z)}{z-z_0} dz + 2\pi i f(z_0)$

By taking $\varepsilon \to 0$ and from (ii), we have $f(z_0)=0$.